On System Dimensionality for the Emergence of Chaos and Random Walks on Infinite Networks

Cristinel Ababei
Milwaukee WI, USA, June 2015

***Abstract*—In this note, we establish a link between the condition for a random walk to be transient on lattices and for a continuous nonlinear system to exhibit chaotic behavior. The link is the number three. A simple random walk is transient on lattices with dimensionality of at least three. Nonlinear dynamical systems can only have chaotic behavior if their dimension is at least three.**

Keywords*—*transient random walk; emergence of chaotic behavior; three dimensional.

I. Minimum Dimensionality for Emergence of Chaotic Behavior

What is chaos? While there is no universally accepted mathematical definition of chaos (and despite that some thought chaos was a fancy word for instability), a very well argued definition is due to Strogatz (p. 323 in [1]): "*Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions*". This definition insists that three ingredients are required. First is the "aperiodic long-term behavior", which says that trajectories do not settle to fixed points, periodic orbits, or quasiperiodic orbits as $x \to \infty$. In other words, chaotic behavior excludes fixed points and periodic behavior. Second, "deterministic" means that there are no noisy or random inputs or parameters in the system. And third, "sensitive dependence on initial conditions" suggests exponentially fast separation or divergence of nearby trajectories (i.e., system has positive Liapunov exponent).

While the above definition conveys well the idea of "extremely erratic behavior", one interesting aspect about it is that it does not say anything about the system dimensionality. Discrete chaotic systems, such as the 1D logistic map, can produce chaotic behavior whatever their dimensionality. Finite-dimensional linear systems are never chaotic; for a dynamical system to display chaotic behavior, it has to be either nonlinear or infinite-dimensional whatever that means. In contrast, for *continuous* dynamical systems, one finds that chaotic behavior can only arise in systems with three or more dimensions.

For example, the following system of three differential equations, known as the Lorenz equations [2], can manifest chaotic behavior:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y-x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= rx - y - xz \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy - bz \end{cases} \tag{1}$$

where $\sigma, r, b > 0$ are the system parameters. x, y, z, make up the system state and t is time. There are two nonlinearities, given by the quadratic terms xz and xy. Lorenz discovered that this system, for certain range of parameters, has solutions that oscillate irregularly and never exactly repeating, that is, are chaotic. Also, these solutions remained in a bounded region of the phase space. When the trajectories of this system are plotted in three dimensions, they settled onto a *complicated set* called a *strange attractor*. A strange attractor is defined to be an attractor that exhibits sensitive dependence on initial conditions and is often a fractal set with a fractional dimension between 2 and 3 [1]. A fractal set, an "infinite complex of surfaces", is a set of points with zero volume but infinite surface area; the motion of trajectories on it is aperiodic and sensitive to minute changes in the initial conditions. A strange attractor is not a point or curve or a surface, it is a more complex structure. Fig. 1 shows the visualization of the Lorenz attractor for a particular set of parameter values and initial conditions.

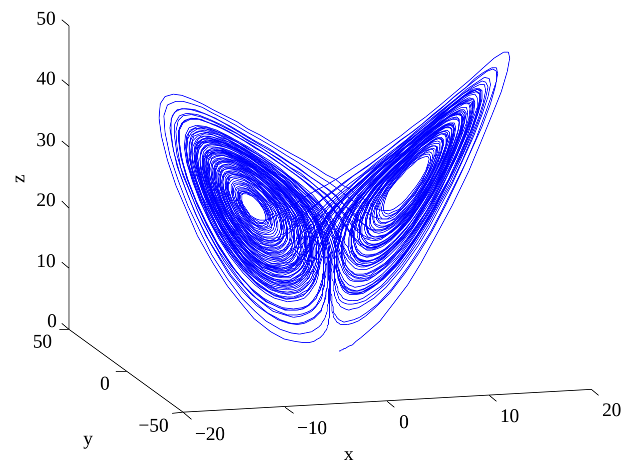

Figure 1. The Lorenz attractor for parameter values $\sigma = 10, r = 28, b = 8/3$ and initial conditions $x = 1, y = 1, z = 1$.

Coming back to the aspect of system dimensionality, the Poincaré-Bendixson theorem, a central result in nonlinear dynamics, says that the dynamical possibilities in the phase plane are very limited in that a trajectory restricted to a closed bounded region with no fixed point will eventually approach a closed orbit [1], [3]. The theorem states that a two-dimensional differential equation has very regular behavior. It implies that chaos can never occur in the phase plane. However, in higher dimensional systems, $n \geq 3$, the theorem does not apply anymore. In such cases, trajectories may wonder around indefinitely within a bounded region and sometimes are attracted to a complex geometric object, the strange attractor, rather than settling down to a fixed point or closed orbit. The Poincaré-Bendixson theorem shows that a strange attractor can only arise in three or more dimensions. It is only in three or higher dimensions where period-doubling bifurcations of limit

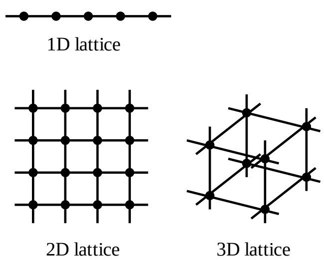

Figure 2. Illustration of 1-, 2-, and 3-dimensional lattices.

cycles can occur to lead to chaos, because the limit cycle needs room to avoid crossing itself [1]!

II. Random Walks on Infinite Networks

One of the first chance processes studied in probability was the *random walk* or *drunkard's walk* . An example of a random walk may be described as follows [4]: A man walks along a 5-block stretch on Madison Avenue. He starts at corner x and, with probability 1/2, walks one block to the right and, with probability 1/2, walks one block to the left; when he comes to the next corner he again randomly chooses his direction along Madison Avenue. He continues until he reaches corner 5, which is home, or corner 0, which is a bar. If he reaches either home or the bar, he stays there. The problem to pose is to find the probability $p(x)$ that the man, starting at corner x, will reach home before reaching the bar.

This problem can be generalized to other types of *street networks*. In 1921 George Polya investigated random walks on certain infinite graphs, which are commonly referred to as lattices [5]. To construct a d-dimensional lattice, we take as vertices those points $(x_1, ..., x_d)$ of $\mathbf{R}^d$ all of whose coordinates are integers, and we join each vertex by an undirected line segment to each of its $2d$ nearest neighbors. These connecting segments, which represent the edges of our graph, each have unit length and run parallel to one of the coordinate axes of $\mathbf{R}^d$. We denote this d-dimensional lattice by $\mathbf{Z}^d$ and its origin $(0, 0, ..., 0)$ by $\mathbf{0}$.

A *simple random walk* in d dimensions can be defined as the walk of a point that starts from some vertex and walks at random by choosing any of the $2d$ edges leading out of that vertex with probability $\frac{1}{2d}$. Obviously, when $d = 1$, our lattice is just an infinite line divided into segments of length one. The random walk, then, is the example discussed at the beginning of this section. When $d = 2$, our lattice looks like a two dimensional infinite regular mesh. When $d = 3$, the lattice can be visualized in space as an infinite 3D regular mesh as shown in Fig. 2.

The question that Polya posed amounts to this: "Is the wandering point certain to return to its starting point during the course of its wanderings?" If so, then, the walk is called *recurrent* and if not, that is, there is a positive probability that the point will never return to its starting point, case in which the walk is called *transient*. The problem of determining recurrence or transience of a random walk is called the *type problem*. A graphical illustration of the transient random walk in a 3-dimensional lattice is shown in Fig. 3. In [5], Polya proved the following theorem: *Simple random walk on a d-dimensional lattice is recurrent for $d = 1, 2$ and transient for $d \geq 3$.*

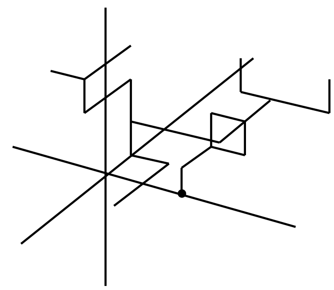

Figure 3. Random walk on 3-dimensional lattice is transient, i.e., the probability that the point never returns to its starting point is greater than zero, $p_{esc} > 0$.

An alternative proof of this theorem was presented by Doyle and Snell in their wonderfully written book [4]. Doyle and Snell presented an interpretation of Polya's theorem in terms of electric networks; they cleverly exploited the connections between questions about a random walk on a graph and questions about electric currents in a corresponding network of resistors. They constructed their proof of Polya's theorem by showing the resistance of the corresponding electric network is infinite in 1 and 2 dimensions - in contrast with the 3 dimensional case, where the resistance is finite, which allows current to flow to infinity.

III. Discussion

By now, the reader probably noticed our fascination with the number three. We summarize our discussion observing the commonality between the two topics discussed in the previous two sections. On one hand, a simple random walk is transient on lattices with dimensionality of at least three. On the other hand, chaotic behavior can emerge in continuous nonlinear dynamical systems whose dimensionality is at least three. Is there any relation between the two?

Can the proof of the transient property of random walks on 3D lattices aid in explaining the need for three dimensions in chaotic behaviors? Can 3D nonlinear systems serve "as true as practically possible" random number generators in computers? Can we talk about a relationship between electric networks and chaos? Is there a connection between the current that can flow to infinity in 3D electric networks and the need for a system to be *dissipative* to be able to have chaotic behavior?

References

[1] Steven H. Strogatz, Nonlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering, Westview Press, 1994.

[2] E.N. Lorenz, "Deterministic Nonperiodic Flow," *J. Atmos. Sci.*, vol. 20, pp. 130-141, 1963.

[3] I. Bendixson, "Sur les courbes définies par des équations différentielles," *Acta Mathematica*, Springer Netherlands, vol. 24, no. 1, pp. 1-88, 1901.

[4] Peter G. Doyle and J. Laurie Snell, Random Walks on Electric Networks, The Mathematical Associaton of America, 1984.

[5] G. Polya, "Uber eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz," *Mathematische Annalen*, vol. 84, pp. 149-160, 1921.